Eco 362: Economic Growth
Fall 2013
Midterm Solutions

Question 1:
a) Assumptions:

- The production function is given by $Y_t = K_t^{\alpha} (e_t L_t)^{1-\alpha}, 0 < \alpha < 1$
- $I_t = \gamma Y_t$ (for γ constant)
- $L_{t+1} = (1+n) L_t$ n assumed constant
- $e_{t+1} = (1+\hat{e}) e_t$ $\hat{e}$ assumed constant

We start with the capital accumulation equation

$$\begin{aligned}
K_{t+1} &= (1-\delta) K_t + I_t \\
K_{t+1} &= (1-\delta) K_t + \gamma Y_t \\
K_{t+1} &= (1-\delta) K_t + \gamma K_t^{\alpha} (e_t L_t)^{1-\alpha} \\
\frac{K_{t+1}}{e_{t+1} L_{t+1}} &= \frac{(1-\delta) K_t + \gamma K_t^{\alpha} (e_t L_t)^{1-\alpha}}{e_{t+1} L_{t+1}} \\
k_{t+1} &= \frac{(1-\delta) K_t + \gamma K_t^{\alpha} (e_t L_t)^{1-\alpha}}{(1+\hat{e}) e_t (1+n) L_t} \\
k_{t+1} &= \frac{1}{(1+\hat{e})(1+n)} \left[(1-\delta) \frac{K_t}{e_t L_t} + \gamma \frac{K_t^{\alpha} (e_t L_t)^{1-\alpha}}{e_t L_t} \right] \\
k_{t+1} &= \frac{(1-\delta) k_t + \gamma k_t^{\alpha}}{(1+\hat{e})(1+n)}
\end{aligned}$$

b) The growth rate of per capita income:

$$\begin{aligned}
y &= \frac{Y}{L} = \frac{K^{\alpha} (eL)^{1-\alpha}}{L} \\
&= \left(\frac{K}{eL} \right)^{\alpha} \frac{eL}{L} \\
&= e k^{\alpha}
\end{aligned}$$

taking logs on both sides

$$\ln y = \ln e + \alpha \ln k$$

Taking the derivative with respect to time

$$\frac{\frac{dy}{dt}}{y} = \frac{\frac{de}{dt}}{e} + \alpha \frac{\frac{dk}{dt}}{k}$$

The growth rate of $y =$ the growth rate of $e + \alpha$(the growth rate of k). In the long run, the growth rate of $k = 0$ as it is in steady state so the growth rate of per capita income is just the growth rate of productivity $(\hat{e})$. In the short run, it may be higher or lower depending on if we are above or below steady state.

The growth rate of Total GDP:

$$\begin{aligned} Y &= K^{\alpha}(eL)^{1-\alpha} \\ &= \left(\frac{K}{eL}\right)^{\alpha} eL \\ &= eLk^{\alpha} \end{aligned}$$

taking logs on both sides

$$\ln Y = \ln e + \ln(L) + \alpha \ln k$$

Taking the derivative with respect to time

$$\frac{\frac{dY}{dt}}{y} = \frac{\frac{de}{dt}}{e} + \frac{\frac{dL}{dt}}{L} + \alpha\frac{\frac{dk}{dt}}{k}$$

The growth rate of $Y =$ the growth rate of e +the growth rate of $L + \alpha$(the growth rate of k). In the long run, the growth rate of $k = 0$ as it is in steady state so the growth rate of GDP is just $(\hat{e} + n)$. In the short run, it may be higher or lower depending on if we are above or below steady state.

c)

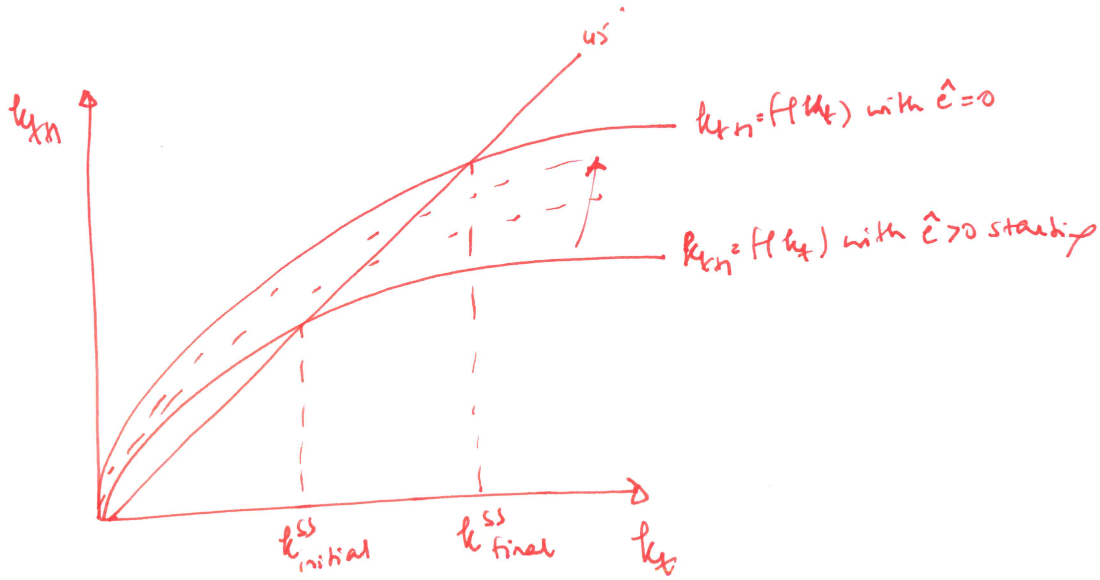
The economy is moving gradually from a positive productivity growth rate $(\hat{e}_{old})$ to a zero productivity growth rate $(\hat{e}_{new} = 0)$.

Effects on $\tilde{k}$:

First let's start by discussing the effects on $\tilde{k}$ i.e. the capital to effective labor ratio, $\left(k_t = \frac{K_t}{e_t L_t}\right)$. The equation that governs the evolution of k is given by

$$k_{t+1} = \frac{(1-\delta) k_t + \gamma k_t^{\alpha}}{(1+n)(1+\hat{e})}$$

Consider a country starting out in steady state $\left(i.e. \text{ at } k_{old}^{SS}\right)$. An increase in $\hat{e}$ will shift the line that shows the relationship between k_{t+1} and k_t upwards. That is, for every k_t the economy will have a higher k_{t+1} with the lower productivity growth rate than it did with the old productivity growth rate $(\hat{e}_{old})$. The reason is that the population and investment behavior remains the same, but now the country has fewer effective workers in the future for every k_t. This implies that the steady state of the economy gradually moves higher till it reaches it highest value when $\hat{e}_{new} = 0$. The economy is currently at lower k relative to its new steady state and so its k will gradually increase till it reaches the new highest steady state



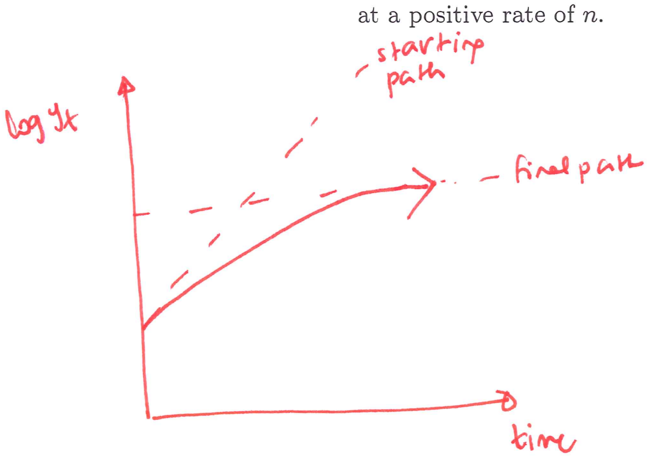
The long run effect on k is level effect of moving from a lower to a higher steady state level. The long run growth rate of k is still zero. In the short run it has positive growth till it reaches its new, higher steady state.

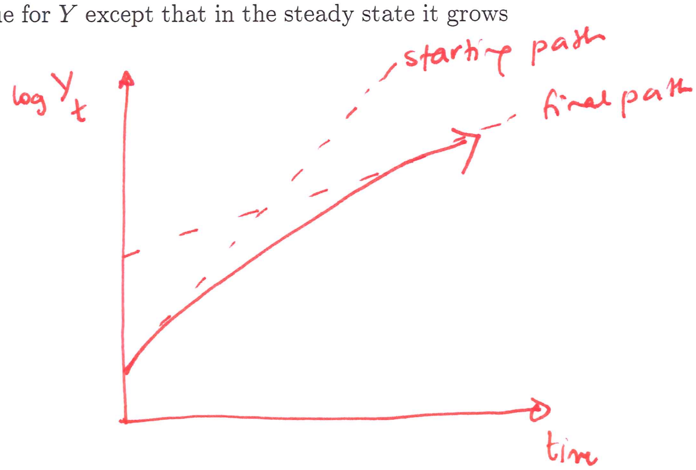
Effects on y :

Per-capita income (y) is a function of k and e_t $(y = e_t k_t^{\alpha})$. It starts out growing along its old steady state growth path (y_{old}^{SS}) and is growing at rate $\hat{e}_{old}$. When the productivity growth rate decreases, k gradually increases till it gets to its new lower steady state. These two effect cause y to grow slower than $\hat{e}_{new}$ till it reaches the new steady state income with zero long run growth.

Effects on Y :

The same effect on y hold true for Y except that in the steady state it grows at a positive rate of n.





2.(a) (7) If we had to empirically estimate β from the data, how would we do it? i.e. what does β correspond to in the data? Explain your reasoning.

Solution: β is the share of human capital income in total income. each unit of human capital earns its marginal product

$$MPh = w_h = \frac{\partial Y}{\partial h} = \beta \frac{Y_t}{h_t}$$

The total income going to skilled labor is

$$w_h * h = \beta Y_t$$

so the share of total income

$$\frac{w_h * h}{Y_t} = \beta$$

Alternatively building off from what we did in the problem set

$$Y_t = K_t^{\alpha}\left(h_t^{\gamma} L_t^{1-\gamma}\right)^{1-\alpha}$$

In this case you would calculate γ as the share of **skilled** labor income in total **wage** income. We know from the Solow model that $(1-\alpha)$ is the **wage** share of income in **total** income So $\beta = \gamma(1-\alpha)$ would be calculated by multiplying the two values obtained from the data. See the problem set for more details and an explicit calculation.

(b)The price of capital

$$r = MPK = \alpha K_t^{\alpha-1} h_t^{\beta} L_t^{1-\alpha-\beta}$$

The Marginal product of physical capital depends on the amount of human capital. In particular, the higher the human capital stock the higher the MPK for the same K, L. Consider

$$\begin{aligned}
\frac{r_R}{r_{TS}} &= \frac{\alpha}{\alpha}\left(\frac{K_R}{K_{TS}}\right)^{\alpha-1}\left(\frac{h_R}{h_{TS}}\right)^{\beta}\left(\frac{L_R}{L_{TS}}\right)^{1-\alpha-\beta} \\
&= \left(\frac{K_R}{K_{TS}}\right)^{\alpha-1}\left(\frac{h_R}{h_{TS}}\right)^{\beta}
\end{aligned}$$

Notice that

$$\begin{aligned}
\frac{r_R}{r_{TS}} &> 1 \\
&\Rightarrow \left(\frac{K_R}{K_{TS}}\right)^{1-\alpha} < \left(\frac{h_R}{h_{TS}}\right)^{\beta}
\end{aligned}$$

Rohan can have a higher price of capital as long as it has a higher human capital stock too. It is high enough so that physical capital is relatively scarce and so has a higher marginal product. For the price of capital to fall because of diminishing marginal product, capital has to become relatively abundant relative to the other factors. For Rohan if human capital is very high then physical capital becomes relatively scarce.

3. (a) You were graded according to the economic reasoning of the effect you stated in part a (i). See problem set for example on one common interpretation – that the investment rate changes.

(b) (10) Suppose you find that the price of capital rose after the shutdown. What is a reasonable explanation for this in the context of the basic Solow model? Explain your reasoning clearly and precisely.

Solution: The price of capital is the MPK with competitive markets. In this model, if we assume that A stays the same with the shutdown, the only way the MPK can rise is if the capital-labor ratio falls below what it currently is at. With diminishing marginal product, for capital to get more productive, it has to get relatively more scarce. This means that the shutdown has the effect of lowering the amount of capital relative to labor in the US. .

4. Short answer questions:

(a) (10) See article

(b) (10) The Solow model predicts that poor countries grow faster than rich *as long as they are on the same path.* More specifically, the Solow model predicts conditional convergence - poor countries grow faster than rich conditional on their parameters.

This is due to diminishing marginal product of capital. When countries are made "as if" they are on the same path, then countries will be poorer if they have a lower $k = \frac{K}{L}$. As the amount of capital per worker rises, because of diminishing marginal product growth slows down.

In reality, if we look at unconditional convergence, it is possible that poor countries may grow slower than rich if they are on a much lower steady state. All the Solow model says is that you grow faster if you are further away from your own steady state.

(c) (7) A politician claim that increasing the investment rates by 10% will increase incomes by 10%. Do you agree or not? Explain your reasoning.

Disagree in the context of the Solow model. Increasing investment rates will increase the fraction of income going towards Investment. However, the extra machines will be subject to diminishing marginal product (DMP). This means that the output that the machines generate will be relatively smaller as k rises due to DMP. You can see this by looking at

$$\frac{y}{k} = \frac{Ak^{\alpha}}{k} = Ak^{\alpha-1}$$

As k rises, then $\frac{y}{k}$ falls. Per capita income is related capital accumulation through the investment channel (γy) As k rises, the depreciation forces (n, δ) remain proportional while the investment force loses group because of DMP. This is why a 10% increase in γ will result in a less than 10% increase in incomes. Note you can see this directly from

$$y^{ss} = \left(\frac{\gamma A}{n+\delta}\right)^{\frac{\alpha}{1-\alpha}}$$